

# The homotopical dimension of random 2-complexes

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## Abstract

Stochastic algebraic topology aims at studying random or partly known spaces which typically arise in applications as configuration spaces of large systems. In this paper we study the Linial–Meshulam model of random two-dimensional complexes. We prove that if the probability parameter  $p$  satisfies  $p \ll n^{-1-\epsilon}$ , where  $\epsilon > 0$  is arbitrary and independent of  $n$ , then a random 2-complex  $Y$  is homotopically one dimensional with probability tending to 1 as  $n \rightarrow \infty$ . More precisely, we show that under this assumption on  $p$ , the complex  $Y$  can be collapsed to a graph in finitely many steps. It is known that the homotopical dimension of  $Y$  is equal to 2 for  $p > 3n^{-1}$ .

## 1 Introduction

Since its inception in 1959 by Erdős and Rényi [ER60], the theory of random graphs has developed into a rapidly growing and widely applicable branch of discrete mathematics, bringing together ideas from graph theory, combinatorics, and probability theory. In one model, a random graph is a subgraph  $\Gamma$  of a complete graph on  $n$  vertices such that every edge of the complete graph is included in  $\Gamma$  with probability  $p$ , independently of the other edges. One is interested in probabilistic features of  $\Gamma$  and their dependence on  $p$  when  $n$  is large. Here  $0 < p < 1$  is a probability parameter which in general may depend on  $n$ . The theory of random graphs [AS00, Bol08, JLR00] offers many spectacular results and predictions, which play an essential role in various engineering and computer science applications. Random graphs also serve within mathematics as accessible models for other, more complex random structures.

Higher dimensional analogs of the aforementioned Erdős–Rényi model were recently suggested and studied by Linial–Meshulam in [LM06], and Meshulam–Wallach in [MW09]. In these models, one generates a random  $d$ -dimensional simplicial complex  $Y$  by considering the full  $d$ -dimensional skeleton of the simplex  $\Delta_n$  on vertices  $\{1, \dots, n\}$  and retaining  $d$ -dimensional faces independently with probability  $p$ . Note that in this construction  $Y$  contains the  $(d-1)$ -dimensional skeleton of  $\Delta_n$ . The work of Linial–Meshulam and Meshulam–Wallach provides threshold functions for the vanishing of the  $(d-1)$ -st homology groups of random complexes with coefficients in a finite abelian group. Threshold functions for the vanishing of the  $d$ -th homology groups were subsequently studied by Kozlov [Koz09].

In this paper, we focus on 2-dimensional random complexes. The corresponding probability space  $G(\Delta_n^{(2)}, p)$  of the Linial–Meshulam model is defined as follows. Let  $\Delta_n$  denote the  $(n-1)$ -dimensional simplex with vertices  $\{1, 2, \dots, n\}$ . Then  $G(\Delta_n^{(2)}, p)$  denotes the set of all 2-dimensional subcomplexes

$$\Delta_n^{(1)} \subset Y \subset \Delta_n^{(2)},$$

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containing the one-dimensional skeleton  $\Delta_n^{(1)}$ . The probability function  $\mathbb{P} : G(\Delta_n^{(2)}, p) \rightarrow \mathbf{R}$  is given by the formula

$$\mathbb{P}(Y) = p^{f(Y)}(1-p)^{\binom{n}{3}-f(Y)}, \quad Y \in G(\Delta_n^{(2)}, p),$$

where  $f(Y)$  denotes the number of faces in  $Y$ . In other words, each of the 2-dimensional simplexes of  $\Delta_n^{(2)}$  is included in a random 2-complex  $Y$  with probability  $p$ , independently of the other 2-simplexes. As in the case of random graphs,  $0 < p < 1$  is a probability parameter which may depend on  $n$ . When  $n$  grows, the model  $G(\Delta_n^{(2)}, p)$  includes all finite 2-dimensional complexes containing the full 1-skeleton  $\Delta_n^{(1)}$ ; however, the likelihood of various topological phenomena is dependent on the value of  $p$ . The theory of deterministic 2-complexes itself is a rich and active field of current research with many challenging open questions, see [HMS93].

The fundamental group of a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$  was investigated by Babson, Hoffman, and Kahle [BHK08]. They showed that for  $p \gg n^{-1/2} \cdot (3 \log n)^{1/2}$ , the group  $\pi_1(Y)$  vanishes asymptotically almost surely (i.e., the probability that  $\pi_1(Y)$  is trivial tends to 1 as  $n \rightarrow \infty$ ). For  $p \ll n^{-1/2-\epsilon}$ , these authors use notions of negative curvature due to Gromov to study the nontriviality and hyperbolicity of  $\pi_1(Y)$ .

In this paper, we show that for  $p \ll n^{-1-\epsilon}$  a random 2-complex  $Y$  is homotopically 1-dimensional, a.a.s.<sup>1</sup> More precisely, we show that  $Y$  can be collapsed to a graph in finitely many steps. This implies that  $Y$  has a free fundamental group and vanishing 2-dimensional homology. Note that the vanishing of 2-dimensional homology in this range of  $p$  also follows from a result of Kozlov [Koz09]. In [CFK10], it is shown that for  $p > 3/n$ , the homology group  $H_2(Y; \mathbf{Z})$  is nontrivial with probability tending to 1; see also [Koz09]. Thus, for  $p > 3/n$ , the random 2-complex  $Y$  is homotopically two-dimensional a.a.s.

Our main result is as follows:

**Theorem 1.** (a) If for some  $k \geq 1$  the probability parameter  $p$  satisfies<sup>2</sup>

$$p \ll n^{-1-\frac{2}{k+1}},$$

then a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$  is collapsible to a graph in at most  $k$  steps, asymptotically almost surely (a.a.s.). (b) If for some  $k \geq 1$  the probability parameter  $p$  satisfies

$$p \gg n^{-1-\frac{1}{3 \cdot 2^{k-1}-1}},$$

then  $Y$  is not collapsible to a graph in  $k$  or fewer steps, a.a.s.

Loosely speaking, Theorem 1 combines with previously known results to suggest that a random 2-complex with vanishing 2-dimensional homology is homotopically one-dimensional.

Theorem 1 implies:

**Corollary 2.** If for some  $k \geq 1$  the probability parameter  $p$  satisfies

$$p \ll n^{-1-\frac{2}{k+1}}$$

then the fundamental group  $\pi_1(Y)$  of a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$  is free and  $H_2(Y; \mathbf{Z}) = 0$ , a.a.s.

The proof of Theorem 1 is given at the very end of the paper. A key role is played by Theorem 13, which states that there exists a finite list of forbidden 2-complexes  $\mathcal{L}_{k,r}$  with  $k \geq 0$  and  $r \geq 2$ , such that an arbitrary 2-complex of degree at most  $r$  (see below) is collapsible to a graph in  $k$  steps if and only if it does not contain any of the 2-complexes from  $\mathcal{L}_{k,r}$ . This allows us to reduce the collapsibility problem to the containment problem for random complexes which was studied in [CFK10].

<sup>1</sup>We use the abbreviation a.a.s. for the phrase “asymptotically almost surely”.

<sup>2</sup>Recall that the symbol  $a_n \ll b_n$  means that  $a_n > 0$  and  $a_n/b_n \rightarrow 0$  as  $n \rightarrow \infty$ .

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## 2 Collapsibility of a 2-complex to a graph

### 2.1 Basic definitions

Let  $Y$  be a finite 2-dimensional simplicial complex. An edge of  $Y$  is called *free* if it is included in exactly one 2-simplex.

The *boundary*  $\partial Y$  is defined as the union of free edges. We say that a 2-complex  $Y$  is *closed* if  $\partial Y = \emptyset$ .

A 2-complex  $Y$  is called *pure* if every maximal simplex is 2-dimensional. By the *pure part* of a 2-complex we mean the maximal pure subcomplex, i.e. the union of all 2-simplexes.

Let  $Y$  be a simplicial 2-complex and let  $\sigma$  and  $\tau$  be two 2-simplexes of  $Y$ . We say that  $\sigma$  and  $\tau$  are adjacent if they intersect in an edge. The *distance* between  $\sigma$  and  $\tau$ ,  $d_Y(\sigma, \tau)$ , is the minimal integer  $k$  such that there exists a sequence of 2-simplexes  $\sigma = \sigma_0, \sigma_1, \dots, \sigma_k = \tau$  with the property that  $\sigma_i$  is adjacent to  $\sigma_{i+1}$  for every  $0 \leq i < k$ . (If no such sequence exists then  $d_Y(\sigma, \tau) = \infty$ .) The *diameter*  $\text{diam}(Y)$  is defined as the maximal value of  $d_Y(\sigma, \tau)$  taken over pairs of 2-simplexes of  $Y$ .

A simplicial 2-complex is *strongly connected* if it has a finite diameter.

A simplicial 2-complex has *degree*  $\leq r$  if every edge is incident to at most  $r$  2-simplexes.

A *pseudo-surface* is a finite, pure, strongly connected 2-dimensional simplicial complex of degree at most 2 (i.e., every edge is included in at most two 2-simplexes).

More generally, for an integer  $r > 0$ , an *r-pseudo-surface* is a finite, pure, strongly connected 2-dimensional simplicial complex of degree at most  $r$ .

### 2.2 Simplicial collapse

Let  $Y$  be a 2-complex. A 2-simplex of  $Y$  is called *free* if at least one of its edges is free. Let  $\sigma_1, \dots, \sigma_k$  be all free 2-simplexes in  $Y$ , and let  $e_1, \dots, e_k$  be free edges with  $e_i \subset \sigma_i$ . We say that the complex

$$Y' = Y - \bigcup_{i=1}^k \text{int}(\sigma_i) - \bigcup_{i=1}^k \text{int}(e_i)$$

is obtained from  $Y$  by collapsing all free 2-simplexes. Clearly  $Y' \subset Y$  is a deformation retract. The operation  $Y \searrow Y'$  is called a *simplicial collapse*. Note that  $Y'$  is not uniquely determined if one of the free simplexes of  $Y$  has two free edges; however the pure part of  $Y'$  (i.e. the union of 2-simplexes of  $Y'$ ) is uniquely determined.

This process can be iterated  $Y' \searrow Y'', Y'' \searrow Y''',$  etc. We denote  $Y = Y^{(0)}, Y' = Y^{(1)}, Y'' = Y^{(2)}$  etc. The sequence of subcomplexes  $Y^{(0)} \supset Y^{(1)} \supset Y^{(2)} \supset \dots$  is decreasing and there are two possibilities: either (a) for some  $k$ , the complex  $Y^{(k)}$  is one-dimensional (a graph), or (b) for some  $k$ , the complex  $Y^{(k)}$  is 2-dimensional and closed, i.e.,  $\partial Y^{(k)} = \emptyset$ .

**Definition 3.** We say that  $Y$  is collapsible to a graph in at most  $k$  steps if  $Y^{(k)}$  is a graph. We say that  $Y$  is collapsible to a graph in  $k$  steps if  $Y^{(k)}$  is a graph and  $\dim Y^{(k-1)} = 2$ .

Observe that if  $Y$  is collapsible to a graph in at most  $k$  steps then any simplicial subcomplex  $S \subset Y$  is also collapsible to a graph in at most  $k$  steps. At each step one removes the free triangles in  $Y^{(i)}$  which belong to  $S$ .

Let  $Y$  be a 2-complex, and consider the sequence of collapses

$$Y^{(0)} \searrow Y^{(1)} \searrow Y^{(2)} \searrow \dots \searrow Y^{(k)} \searrow \dots$$

For a 2-simplex  $\sigma \in Y$  define

$$D_Y(\sigma) = \sup\{i; \sigma \subset Y^{(i)}\} \in \{0, 1, \dots, \infty\}.$$

A 2-simplex  $\sigma$  is free if and only if  $D_Y(\sigma) = 0$ .

A 2-complex  $Y$  is collapsible to a graph in at most  $k + 1$  steps if and only if  $D_Y(\sigma) \leq k$  for any 2-simplex  $\sigma$ . If after performing several collapses  $Y^{(0)} \searrow Y^{(1)} \searrow Y^{(2)} \searrow \dots$  we obtain a subcomplex  $Y^{(r)} \subset Y$  with empty boundary  $\partial Y^{(r)} = \emptyset$ , then  $Y^{(r)} = Y^{(r+1)} = Y^{(r+2)} = \dots$  and  $D_Y(\sigma) = \infty$  for any simplex  $\sigma$  in  $Y^{(r)}$ .

**Lemma 4.** *Let  $\sigma$  be a 2-simplex with  $D_Y(\sigma) = k$  where  $0 < k < \infty$ . Then one of the edges  $e$  of  $\sigma$  has the following property: for any 2-simplex  $\sigma'$  of  $Y$  which is incident to  $e$  and distinct from  $\sigma$  one has  $D_Y(\sigma') < k$  and there exists a 2-simplex  $\sigma''$  incident to  $e$  and distinct from  $\sigma$  such that  $D_Y(\sigma'') = k - 1$ .*

*Proof.* Since  $D_Y(\sigma) = k$ , we know that after  $k$  collapses an edge  $e$  of  $\sigma$  becomes free. All other simplexes  $\sigma'$  of  $Y$  incident to  $e$  must have been eliminated in previous steps, i.e., they satisfy  $D_Y(\sigma') < k$ . At least one of these simplexes  $\sigma'$  must have been eliminated in step  $k - 1$  since otherwise  $\sigma$  would have become free earlier.  $\square$

**Lemma 5.** *If  $Z \subset Y$  is a subcomplex and  $\sigma \subset Z$  is a 2-simplex, then*

$$D_Z(\sigma) \leq D_Y(\sigma).$$

*Proof.* If a 2-simplex belongs to  $Z$  and is not free in  $Z$  then it is not free in  $Y$ . This implies that  $Z' \subset Y'$  and therefore  $Z^{(i)} \subset Y^{(i)}$  for any  $i \geq 1$ . Thus, the maximal  $i$  such that  $\sigma$  is contained in  $Z^{(i)}$  is less than or equal to the maximal  $i$  such that  $\sigma$  is contained in  $Y$ , which implies the statement of the Lemma.  $\square$

## 2.3 $\sigma$ -accessible boundary

**Definition 6.** *Let  $Y$  be a 2-complex and let  $\sigma, \tau$  be two 2-simplexes of  $Y$  with  $D_Y(\tau) = 0$  and  $D_Y(\sigma) = k \geq 1$ . A collapsing path from  $\tau$  to  $\sigma$  is a sequence of 2-simplexes  $\tau = \sigma_0, \sigma_1, \dots, \sigma_{k-1}, \sigma_k = \sigma$  such that  $D_Y(\sigma_i) = i$  and each pair  $\sigma_i$  and  $\sigma_{i+1}$  has a common edge, where  $i = 0, \dots, k - 1$ .*

In a collapsing path, the initial simplex  $\sigma_0 = \tau$  is a free simplex, and hence at least one of its edges belongs to the boundary  $\partial Y$ .

**Definition 7.** *Given a 2-simplex  $\sigma$ , we denote by  $A_Y(\sigma) \subset \partial Y$  the union of the edges in  $\sigma_0 \cap \partial Y$  which can appear in a collapsing path  $\sigma_0, \sigma_1, \dots, \sigma_k$  ending at  $\sigma$ . We call  $A_Y(\sigma)$  the  $\sigma$ -accessible part of the boundary.*

In Definition 7, clearly  $k = D_Y(\sigma)$ . Note that  $A_Y(\sigma) \neq \emptyset$  if and only if  $D_Y(\sigma) < \infty$ .

**Definition 8.** *Let  $\sigma$  be a 2-simplex of  $Y$  with  $D_Y(\sigma) \geq 1$ . For an edge  $e$  of  $\sigma$  define*

$$A_Y(\sigma, e) \subset A_Y(\sigma)$$

*as the set of all edges  $e'$  of the boundary  $\partial Y$  with the property that there exists a collapsing path  $\sigma_0, \sigma_1, \dots, \sigma_k = \sigma$  such that  $e'$  is an edge of  $\sigma_0$  and  $e = \sigma_{k-1} \cap \sigma_k$ .*

If  $e_1, e_2, e_3$  are the edges of  $\sigma$  then  $A_Y(\sigma) = \cup_{i=1}^3 A_Y(\sigma, e_i)$  and the sets  $A_Y(\sigma, e_i)$  need not be mutually disjoint.

**Lemma 9.** *Let  $\sigma$  and  $\sigma'$  be adjacent 2-simplexes of  $Z$  with*

$$D_Z(\sigma) = D_Z(\sigma') + 1.$$

*Assume that any collapsing path in  $Z$  ending at  $\sigma$  passes through the edge  $e = \sigma \cap \sigma'$ . If  $Z$  is embedded as a subcomplex  $Z \subset Y$  and*

$$D_Z(\sigma') < D_Y(\sigma'),$$

*then*

$$D_Z(\sigma) < D_Y(\sigma).$$

*Proof.* Let  $k = D_Z(\sigma') = D_Z(\sigma) - 1$ . We must show that  $D_Y(\sigma) \geq k + 2$ . First we claim that the edge  $e$  may become free only after at least  $k + 2$  collapses in  $Y$ . Assume it is free in  $Y$  after  $k + 1$  collapses. By assumption,  $D_Y(\sigma') \geq k + 1$ . Hence the edge  $e$  can only be free after  $k + 1$  collapses in  $Y$  if  $\sigma$  has been removed already before, i.e.,  $D_Y(\sigma) \leq k$ . On the other hand, by Lemma 5,  $D_Y(\sigma) \geq D_Z(\sigma) = k + 1$  which leads to a contradiction.

By assumption, the two edges of  $\sigma$  different from  $e$  are not free in  $Z^{(k+1)}$  and hence they are not free in  $Y^{(k+1)}$ . Thus  $D_Y(\sigma) \geq k + 2$  as claimed.  $\square$

Note that the assumption of Lemma 9 that any collapsing path in  $Z$  ending at  $\sigma$  passes through the edge  $e$  is equivalent to  $A_Z(\sigma, e') = \emptyset$  for the two remaining edges  $e' \neq e$  of  $\sigma$ .

**Lemma 10.** *Let  $Z \subset Y$  be a subcomplex. If  $D_Z(\sigma) = D_Y(\sigma)$  for a 2-simplex  $\sigma$  of  $Z$  then there is an edge  $e$  of  $\sigma$  such that*

$$\emptyset \neq A_Z(\sigma, e) \subset A_Y(\sigma, e) \subset \partial Y.$$

*Proof.* Without loss of generality, we may assume that  $Y$  is obtained from  $Z$  by attaching a single 2-simplex.

The proof is by induction on  $k = D_Y(\sigma) = D_Z(\sigma)$ .

In the case  $k = 0$ , there is an edge  $e$  of  $\sigma$  that is free in both  $Z$  and  $Y$ . In particular,  $e \subset \partial Y$ .

We include the case  $k = 1$ . Recall that  $Z' = Z^{(1)}$  denotes the result of the first collapse of  $Z$ ,  $Z \searrow Z'$ . Since  $D_Z(\sigma) = D_Y(\sigma) = 1$ , there is an edge  $e$  of  $\sigma$  that is free in  $Y'$  and hence in  $Z'$ . Then every collapsing path  $\tau, \sigma$  in  $Z$  with  $e = \tau \cap \sigma$  is also a collapsing path in  $Y$ . Hence  $A_Z(\sigma, e) \subset A_Y(\sigma, e)$ .

For the general case, assume that  $D_Y(\sigma) = D_Z(\sigma) = k$ . After  $k$  collapses

$$Z \searrow Z^{(1)} \searrow \dots \searrow Z^{(k)}, \quad Y \searrow Y^{(1)} \searrow \dots \searrow Y^{(k)},$$

the 2-simplex  $\sigma$  is exposed in both  $Z^{(k)}$  and  $Y^{(k)}$ . Thus,  $\sigma$  has a free edge  $e$  in  $Y^{(k)}$  (and hence in  $Z^{(k)}$  as well). Writing  $Z' = Z^{(1)}$  and  $Y' = Y^{(1)}$ , by induction, we have  $\emptyset \neq A_{Z'}(\sigma, e) \subset A_{Y'}(\sigma, e)$  so that any collapsing path  $\sigma_1, \dots, \sigma_k$  from  $\sigma_1 = \sigma' \subset A_{Z'}(\sigma, e)$  to  $\sigma_k = \sigma$  in  $Z'$  is also a collapsing path in  $Y'$ . Note in particular that every edge of  $\sigma'$  that is free in  $Z'$  is also free in  $Y'$ . Consequently, for every free triangle  $\tau$  in  $Z$  which meets  $\sigma'$  in an edge free in  $Z'$ , the collapsing path  $\tau = \sigma_0, \sigma_1, \dots, \sigma_k$  in  $Z$  is a collapsing path in  $Y$ . The result follows.  $\square$

**Corollary 11.** *Let  $Z \subset Y$  be 2-complexes such that for a 2-simplex  $\sigma$  of  $Z$  none of the edges  $e \in A_Z(\sigma) \subset \partial Z$  is free in  $Y$ . Then*

$$D_Z(\sigma) + 1 \leq D_Y(\sigma).$$

*Proof.* For a contradiction, assume that  $D_Y(\sigma) \leq D_Z(\sigma)$ . Then  $D_Y(\sigma) = D_Z(\sigma)$  by Lemma 5. We may now apply Lemma 10 which claims that there is an edge  $e$  of  $\sigma$  for which  $\emptyset \neq A_Z(\sigma, e) \subset A_Y(\sigma, e) \subset \partial Y$ . This contradicts our assumption that no edge in  $A_Z(\sigma)$  lies on the boundary  $\partial Y$ .  $\square$

## 2.4 The list of forbidden $r$ -pseudo-surfaces $\mathcal{L}_{k,r}$

For a pair of integers  $k = 0, 1, \dots$ , and  $r = 2, 3, \dots$  we denote by  $\mathcal{L}_{k,r}$  the set of all isomorphism types of  $r$ -pseudo-surfaces  $S$  with the following properties:

- (a) Each  $S \in \mathcal{L}_{k,r}$  has a specified 2-simplex  $\sigma_*$  (called *the center*).
- (b) If  $\partial S \neq \emptyset$  then  $D_S(\sigma_*) = k$ .
- (c)  $d_S(\sigma_*, \sigma) \leq k$  for any 2-simplex  $\sigma$ .

Note that  $\mathcal{L}_{0,r} = \{S\}$  consists of a single complex  $S = \sigma_*$  (the triangle).

The set  $\mathcal{L}_{1,2}$  consists of the three surfaces shown in Figure 1. Each of the surfaces a, b, c is a union of 4 triangles. The surface c is a tetrahedron, b is a tetrahedron with one face open, and a is a fully flattened tetrahedron.

It is clear that  $\mathcal{L}_{k,r}$  is finite and  $\mathcal{L}_{k,r} \subset \mathcal{L}_{k,r+1}$ .

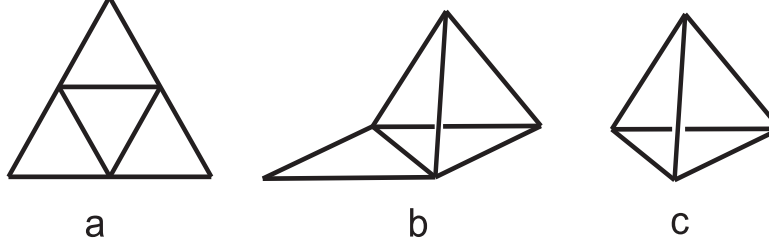


Figure 1: Surfaces  $\mathcal{L}_{1,2}$ .

**Example 12.** Consider the following important family of surfaces  $S_k \in \mathcal{L}_{k,2}$  where  $k = 0, 1, 2, \dots$ . The first surface  $S_0$  is defined as a single triangle  $S_0 = \sigma_*$ . The next surface  $S_1$  is the shown in Figure 1 a. Surfaces  $S_2$  and  $S_3$  are shown in Figure 2. In general, the surface  $S_k$  is obtained from  $S_{k-1}$  by adding a triangle to every edge of the boundary  $\partial S_{k-1}$ . It is clear that for the central triangle  $\sigma_*$  of  $S_k$ , one has  $D_{S_k}(\sigma_*) = k$ . Thus  $S_k$  is not collapsible to a graph in  $k$  steps, but is collapsible in  $k + 1$  steps.

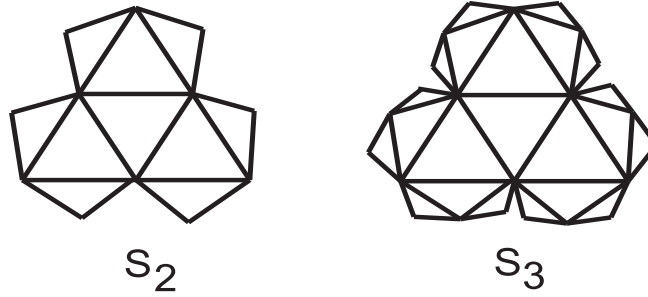


Figure 2: Surfaces  $S_k \in \mathcal{L}_{k,2}$ .

The following Theorem plays a key role in this paper:

**Theorem 13.** A 2-complex  $Y$  of degree at most  $r \geq 2$  is not collapsible to a graph in  $k$  steps, where  $k = 0, 1, 2, \dots$ , if and only if there is a surface  $S \in \mathcal{L}_{k,r}$  which admits a simplicial embedding  $S \rightarrow Y$ .

In the proof, we will use the following statement:

**Lemma 14.** Let  $Y$  be a finite 2-dimensional simplicial complex of degree at most  $r$  and let  $\sigma$  be a 2-simplex in  $Y$  with  $D_Y(\sigma) = k$ , where  $k = 0, 1, 2, \dots$ . Then there exists a surface  $S \in \mathcal{L}_{k,r}$  and a simplicial embedding  $S \rightarrow Y$  such that the central simplex  $\sigma_*$  of  $S$  is mapped onto  $\sigma$ .

*Proof of Lemma 14.* We will use induction on  $k = D_Y(\sigma)$ . For  $k = 0$ , the statement is obvious. Assume that it is true for all cases with  $D_Y(\sigma) < k$ , and consider the situation when  $D_Y(\sigma) = k > 0$ . If  $Y \searrow Y'$  is the first collapse, then  $\sigma \subset Y'$  and clearly

$$D_{Y'}(\sigma) = k - 1$$

and  $Y'$  has degree at most  $r$ . By the inductive hypothesis, there exists  $S' \in \mathcal{L}_{k-1,r}$  and a simplicial embedding  $S' \rightarrow Y'$ , mapping the central simplex of  $S'$  onto  $\sigma$ .

For each edge  $e$  lying in  $A_{S'}(\sigma)$  choose a 2-simplex  $\sigma_e \subset Y$  as follows. If  $e \subset \partial Y'$ , let  $\sigma_e$  be any free triangle in  $Y$  containing  $e$ . If  $e \not\subset \partial Y'$ , let  $\sigma_e$  be any triangle in  $Y'$  containing  $e$  which is not in  $S'$ ; such  $\sigma_e$  exists since  $e \not\subset \partial Y'$ .

Next we define a subcomplex  $S \subset Y$  as the union

$$S = S' \cup \bigcup_e \sigma_e \subset Y,$$

where  $e$  runs over the edges in  $A_{S'}(\sigma)$ . Note that  $S$  is finite, pure, and strongly connected since  $S'$  is an  $r$ -pseudo-surface. Moreover, the degree of  $S$  is at most  $r$  since it is a subcomplex of  $Y$ . One has  $D_S(\sigma) \geq k$  by Corollary 11. More precisely, we obtain that  $D_S(\sigma) = k$  by Lemma 5. Finally we observe that obviously  $d_S(\sigma, \sigma') \leq k$  for any 2-simplex  $\sigma'$  of  $S$ . Thus,  $S \in \mathcal{L}_{k,r}$ .  $\square$

*Proof of Theorem 13.* Consider the sequence of successive collapses  $Y \searrow Y^{(1)} \searrow Y^{(2)} \searrow Y^{(3)} \searrow \dots$ . We assume that  $Y$  is not collapsible to a graph in  $k$  steps, which implies that there are two possibilities: either (a)  $Y^{(i)} \neq Y^{(i+1)}$  for any  $i < k$ ; or (b) for some  $i < k$ , one has  $\partial Y^{(i)} = \emptyset$ .

In case (a), the complex  $Y$  contains a 2-simplex with  $D_Y(\sigma) = k$  and Lemma 14 gives us an embedding of an  $r$ -pseudo-surface  $S \in \mathcal{L}_{k,r}$  into  $Y$ .

In case (b), we have  $\partial Y^{(i)} = \emptyset$  for some  $i < k$ . Fix a 2-simplex  $\sigma_* \in Y^{(i)}$  and consider distances  $d_{Y^{(i)}}(\sigma_*, \sigma)$  to various 2-simplexes  $\sigma$  of  $Y^{(i)}$ . If all these distances are less than or equal to  $k$ , then  $Y^{(i)}$  belongs to  $\mathcal{L}_{k,r}$  and we are done. If there are simplexes  $\sigma$  such that  $d_{Y^{(i)}}(\sigma_*, \sigma) > k$ , then consider the subcomplex  $Z \subset Y^{(i)}$  defined as the union of all  $\sigma$  with  $d_{Y^{(i)}}(\sigma_*, \sigma) \leq k$ .

Clearly  $Z$  is not collapsible to a graph in  $k$  steps. Therefore, in the sequence of collapses  $Z \searrow Z^{(1)} \searrow Z^{(2)} \searrow Z^{(3)} \searrow \dots$ , we again have either case (a) or (b) as above. In case (a), we apply Lemma 14; and in case (b), we obtain a subcomplex  $S \subset Z$  with  $\partial S = \emptyset$  such that  $d(\sigma_*, \sigma) \leq k$  for any  $\sigma \in S$ . We have  $S \in \mathcal{L}_{k,r}$  in either case, completing the proof.  $\square$

### 3 Collapsibility of a random 2-complex

#### 3.1 The degree sequence

Recall that the degree of an edge  $e$  in a 2-complex is defined as the number of 2-simplexes which contain  $e$ . The degree of an edge in a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$  is an integer in the set  $\{0, 1, \dots, n-2\}$ .

Let  $X_k : G(\Delta_n^{(2)}, p) \rightarrow \mathbf{Z}$  be the random variable counting the number of edges of degree  $k$  in a random 2-complex, where  $k = 0, 1, 2, \dots, n-2$ . A straightforward calculation reveals that

$$\mathbb{E}(X_k) = \binom{n}{2} \binom{n-2}{k} p^k (1-p)^{n-2-k}.$$

The expectation of the number of edges of degree at least  $r$  in a random 2-complex is

$$\sum_{k=r}^{n-2} \mathbb{E}(X_k) \leq n^2 \sum_{k=r}^{n-2} (pn)^k \leq \frac{n^2 (pn)^r}{1-pn}. \quad (1)$$

**Corollary 15.** *The probability that a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$  has an edge of degree at least  $r$  is less than or equal to*

$$\frac{n^{2+r} p^r}{1-pn}.$$

Thus, if

$$p \ll n^{-1-\frac{2}{r}},$$

then a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$  has no edges of degree  $r$  or greater, a.a.s.

*Proof.* This follows from inequality (1) by applying the first moment method, see, for instance, [JLR00].  $\square$

### 3.2 The invariant $\tilde{\mu}(S)$ .

Following [BHK08] and [CFK10], for a 2-complex  $S$  with  $v = v(S)$  vertices and  $f = f(S) > 0$  faces one defines

$$\mu(S) = \frac{v}{f} \in \mathbb{Q},$$

and

$$\tilde{\mu}(S) = \min_{S' \subset S} \mu(S'),$$

where  $S'$  runs over all subcomplexes of  $S$  or, equivalently, over all pure subcomplexes  $S' \subset S$ . Note the following *monotonicity property* of  $\tilde{\mu}$ :

$$\text{if } S \subset T, \text{ then } \tilde{\mu}(S) \geq \tilde{\mu}(T). \quad (2)$$

The invariant  $\tilde{\mu}$  controls embeddability of finite 2-complexes into random 2-complexes as illustrated by the following result.

**Theorem 16** ([CFK10]). *Let  $S$  be a finite simplicial complex.*

- (a) *If  $p \ll n^{-\tilde{\mu}(S)}$ , the probability that  $S$  admits a simplicial embedding into a random 2-complex  $Y \subset G(\Delta_n^{(2)}, p)$  tends to zero as  $n \rightarrow \infty$ ;*
- (b) *If  $p \gg n^{-\tilde{\mu}(S)}$ , the probability that  $S$  admits a simplicial embedding into a random 2-complex  $Y \subset G(\Delta_n^{(2)}, p)$  tends to one as  $n \rightarrow \infty$ .*

**Definition 17.** *A 2-complex  $S$  is called balanced if  $\tilde{\mu}(S) = \mu(S)$ , or, equivalently,  $\mu(S') \geq \mu(S)$  for any subcomplex  $S' \subset S$ .*

Any triangulated surface is balanced, see [CFK10].

**Example 18.** Suppose that a 2-complex  $S$  has a free triangle with two free edges, and that the result  $S'$  of removing this triangle satisfies  $\mu(S') < 1$ . Then  $\mu(S) > \mu(S')$  and  $S$  is unbalanced. Indeed, if  $\mu(S') = v/f$ , where  $v = v(S')$  and  $f = f(S')$ , then  $v < f$  and we have  $\mu(S) = (v+1)/(f+1) > v/f$ . In this way one produces many unbalanced 2-complexes, including 2-disks.

Next, we examine the  $\tilde{\mu}$  invariants of 2-complexes  $S \in \mathcal{L}_{k,r}$ .

**Lemma 19.** *Let  $S$  be a closed 2-complex, i.e.,  $\partial S = \emptyset$ . Then  $\tilde{\mu}(S) \leq 1$ .*

*Proof.* Without loss of generality, we may assume that  $S$  is connected, since otherwise we can apply the following arguments to a connected component of  $S$  and use the monotonicity property (2). Moreover, we may assume that  $S$  is pure, since otherwise we may deal with the maximal pure subcomplex of  $S$  instead of  $S$ .

Suppose first that  $H_2(S; \mathbb{Z}_2) = 0$ . Then by the Euler–Poincaré theorem,  $\chi(S) \leq 1$ , and we have

$$v - e + f = \chi(S) \leq 1, \quad \text{and} \quad 3f \geq 2e,$$

where  $v, e, f$  denote the numbers of vertices, edges and faces in  $S$ . In the latter inequality we used the assumptions that  $S$  is pure and closed. These inequalities imply

$$v - f/2 \leq \chi(S) \leq 1, \quad \text{and} \quad \mu(S) \leq 1/2 + 1/f.$$

Since  $f \geq 4$  we obtain that  $\tilde{\mu}(S) \leq \mu(S) \leq 3/4 < 1$ .

Assume now that  $H_2(S; \mathbb{Z}_2) \neq 0$ . We will show that there is a subcomplex  $S' \subset S$  which is also closed,  $\partial S' = \emptyset$ , and satisfies  $H_2(S'; \mathbb{Z}_2) = \mathbb{Z}_2$ . Indeed, consider a nonzero two-dimensional cycle  $c = \sum_{i \in I} \sigma_i$  with  $\mathbb{Z}_2$  coefficients, where the  $\sigma_i$  are distinct 2-simplexes of  $S$ . Let  $I' \subseteq I$  be the minimal subset of the indexing set  $I$  for which  $c' = \sum_{i \in I'} \sigma_i$  is still a cycle, and let  $S' = \bigcup_{i \in I'} \sigma_i$  be the corresponding subcomplex of  $S$ . Then clearly  $H_2(S'; \mathbb{Z}_2) = \mathbb{Z}_2$  and  $S'$  is closed and pure.

By the Euler–Poincaré theorem,  $\chi(S') \leq 2$ , and we have

$$v' - e' + f' = \chi(S') \leq 2, \quad \text{and} \quad 3f' \geq 2e',$$



where  $v', e', f'$  denote the numbers of vertices, edges and faces in  $S'$ . This gives

$$v' - f'/2 \leq \chi(S') \leq 2,$$

and

$$\mu(S') \leq \frac{1}{2} + \frac{2}{f'}. \quad (3)$$

Since  $f' \geq 4$ , the last inequality gives  $\mu(S') \leq 1$ . Finally, we have  $\tilde{\mu}(S) \leq \mu(S') \leq 1$ .  $\square$

**Lemma 20.** *If  $S \in \mathcal{L}_{k,r}$  for some  $k \geq 0$ ,  $r \geq 2$  then one has*

$$\tilde{\mu}(S) \leq 1 + \frac{2}{k+1}. \quad (4)$$

*Proof.* If  $S$  is closed the result follows from Lemma 19. Assume now that  $\partial S \neq \emptyset$ . Let  $\sigma_*$  be the central simplex of  $S$  and let  $\sigma_0, \sigma_1, \dots, \sigma_k = \sigma_*$  be a collapsing path leading to  $\sigma_*$ . Here  $D_S(\sigma_i) = i$  and  $\sigma_i \cap \sigma_{i+1}$  is an edge, see Definition 7. Then the union  $S' = \cup_{i=0}^k \sigma_i$  is a subcomplex having exactly  $k+1$  faces and at most  $k+3$  vertices. Thus,

$$\mu(S') \leq \frac{k+3}{k+1} = 1 + \frac{2}{k+1},$$

establishing (4).  $\square$

### 3.3 The threshold for $k$ -collapsibility.

**Definition 21.** *Let  $\tilde{\mu}_{k,r}$  denote the largest possible value of the invariant  $\tilde{\mu}(S)$  for  $S$  a forbidden  $r$ -pseudo-surface,*

$$\tilde{\mu}_{k,r} = \max_{S \in \mathcal{L}_{k,r}} \tilde{\mu}(S) \in \mathbb{Q}.$$

For instance, examining the surfaces shown in Figure 1 reveals that  $\tilde{\mu}_{1,2} = 3/2$ .

**Theorem 22.** *Consider a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$ .*

(a) *If for some  $r \geq 2$  and  $k \geq 1$ , one has*

$$p \ll n^{-1-\frac{2}{r+1}} \quad \text{and} \quad p \ll n^{-\tilde{\mu}_{k,r}},$$

*then  $Y$  is collapsible to a graph in at most  $k$  steps, a.a.s.*

(b) *If for some  $r \geq 2$  and  $k \geq 1$ , one has  $p \gg n^{-\tilde{\mu}_{k,r}}$ , then  $Y$  is not collapsible to a graph in  $k$  or fewer steps, a.a.s.*

*Proof.* By Corollary 15, if  $p \ll n^{-1-\frac{2}{r+1}}$ , then a random 2-complex  $Y \in G(\Delta_n^{(2)}, p)$  has degree at most  $r$ , a.a.s. Next, we apply Theorem 13 and examine the embeddability of complexes  $S \in \mathcal{L}_{k,r}$  into  $Y$ . By Theorem 16 (a), if  $p \ll n^{-\tilde{\mu}(S)}$ , then  $S$  does not embed into  $Y$ , a.a.s. Since  $\tilde{\mu}_{k,r} \geq \tilde{\mu}(S)$ , we see that the assumption  $p \ll n^{-\tilde{\mu}_{k,r}}$  implies that no  $S \in \mathcal{L}_{k,r}$  can be embedded into  $Y$ , a.a.s. Thus, by Theorem 13, we see that  $Y$  is collapsible to a graph in  $k$  or fewer steps. This proves part (a).

To prove part (b), we apply Theorem 16 (b) to conclude that if  $p \gg n^{-\tilde{\mu}_{k,r}}$ , then there exists  $S \in \mathcal{L}_{k,r}$  which is embeddable into  $Y$ , a.a.s. This implies that  $Y$  is not collapsible to a graph in at most  $k$  steps, a.a.s.  $\square$

**Example 23.** Consider the surface  $S_k \in \mathcal{L}_{k,2}$  introduced in Example 12. Note that  $S_k \in \mathcal{L}_{k,r}$  for any  $r \geq 2$ . The numbers of vertices  $v_k$  and faces  $f_k$  of  $S_k$  satisfy the recurrence relations

$$v_k = 2 \cdot v_{k-1} \quad \text{and} \quad f_k = v_{k-1} + f_{k-1}. \quad (5)$$

Indeed, viewing  $S_{k-1}$  as a subcomplex of  $S_k$ , we see that all vertices of  $S_{k-1}$  lie on the boundary, and each edge of the boundary of  $S_{k-1}$  adds a vertex to  $S_k$ . This explains the first equation. For the second, note that the number of new triangles in  $S_k$  is equal to the number of edges on  $\partial S_{k-1}$ .

Since  $v_0 = 3$  and  $f_0 = 1$ , solving the recurrence relations (5) yields

$$v_k = 3 \cdot 2^k \quad \text{and} \quad f_k = 3 \cdot 2^k - 2.$$

Consequently,

$$\mu(S_k) = 1 + \frac{1}{3 \cdot 2^{k-1} - 1}.$$

**Lemma 24.** *The surface  $S_k$  is balanced, and hence*

$$\tilde{\mu}(S_k) = \mu(S_k) = 1 + \frac{1}{3 \cdot 2^{k-1} - 1}.$$

*Proof.* Let  $S$  be a pure subcomplex of  $S_k$  with  $v = v(S)$  vertices and  $f = f(S)$  faces. Write  $v = v_k - m$  and  $f = f_k - n$ , where  $v_k$  and  $f_k$  are as above and  $m$  and  $n$  are the number of vertices and faces which are in  $S_k$ , but not in  $S$ . We claim that  $m = v_k - v \leq f_k - f = n$ . This assertion is established by induction.

The case  $k = 0$  is trivial. So assume inductively that for any  $i < k$  and  $S' \subset S_i$  a pure subcomplex, we have  $v(S_i) - v(S') \leq f(S_i) - f(S')$ .

For a pure subcomplex  $S \subset S_k$  as above, let  $S'$  be the pure part of  $S \cap S_{k-1}$ . Then,  $m = m' + m''$  and  $n = n' + n''$ , where  $v(S') = v_{k-1} - m'$ ,  $f(S') = f_{k-1} - n'$ ,  $m''$  is the number of vertices in  $S_k \setminus S_{k-1}$  which are not in  $S$ , and  $n''$  is the number of faces in  $S_k \setminus S_{k-1}$  which are not in  $S$ .

We have  $m' \leq n'$  by induction. Observe that the vertices of  $S_k \setminus S_{k-1}$  are in one-to-one correspondence with the faces of  $S_k \setminus S_{k-1}$ . If such a vertex is not in  $S$ , then the corresponding face cannot be in  $S$  either. Consequently,  $m'' = n''$ , and  $m = m' + m'' \leq n' + n'' = n$ , completing the proof of the claim.

It follows immediately that  $\mu(S) \geq \mu(S_k) = \mu_k$ . Indeed,

$$\frac{v}{f} - \frac{v_k}{f_k} = \frac{v_k - m}{f_k - n} - \frac{v_k}{f_k} = \frac{nv_k - mf_k}{f_k(f_k - n)} = \frac{\mu_k n - m}{f_k - n} \geq \frac{n - m}{f_k - n} \geq 0.$$

Thus,  $S_k$  is balanced. □

From Lemmas 20 and 24 we obtain:

**Corollary 25.** *For any  $r \geq 2$  and  $k \geq 0$ , one has the following inequalities:*

$$1 + \frac{1}{3 \cdot 2^{k-1} - 1} \leq \tilde{\mu}_{k,r} \leq 1 + \frac{2}{k+1}.$$

Note that the obtained upper and lower bounds for  $\tilde{\mu}_{k,r}$  are independent of  $r$ .

We believe that  $\tilde{\mu}_{k,r} = 1 + 1/(3 \cdot 2^{k-1} - 1)$ .

*Proof of Theorem 1.* The main theorem is now an immediate consequence of Theorem 22 and Corollary 25:

(a) Assume that  $p \ll n^{-1-2/(k+1)}$  for some  $k \geq 1$ . According to Corollary 25,  $\tilde{\mu}_{k,r} \leq 1 + 2/(k+1)$ . Choosing  $r = \max(2, k)$ , it then follows from Theorem 22 (a) that  $Y \in G(\Delta_n^{(2)}, p)$  is collapsible to a graph in at most  $k$  steps, a.a.s.

(b) Assume that  $p \gg n^{-1-1/(3 \cdot 2^{k-1} - 1)}$  for some  $k \geq 1$ . Then by Theorem 16 and Lemma 24 the surface  $S_k$  (see Example 12) embeds into  $Y$ , a.a.s. Since  $S_k$  cannot be collapsed to a graph in  $k$  or fewer steps we obtain that  $Y$  is not collapsible to a graph in  $k$  or fewer steps. □

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